Geometric singular perturbation theory: topics

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1 Prerequisites

- Dynamical systems: phase plane analysis, stable/unstable manifolds [18, Chapters 1-3] [local link]; more detail: [12]
- Asymptotics: order symbols, asymptotic expansions [18, Chapter 9] [local link], [11, Section 1.6]

2 Introductory topics

- Geometric concepts in dynamical systems [8, Section 2-4]
- Fenichel's theorems [6], [11, Section 3.1-3.2], [8, Section 5]
- Jumps and transversal intersections [11, Section 6.1], [6, Section 5]

3 Direct applications

- Electrical circuits: Josephson junctions [14]
- Semiconductors: Travelling waves of electrons with different effective mass [17]
- Ecology: cyclic behaviour in complex predator-prey systems [16]
- Ecology: rapid evolution in predator-prey interactions [1]
- Economics & environment: catastrophic transitions in an artificial 'Wonderland' [13]
- Epidemiology: behavioural changes due to epidemics [15]
- Gas/fluid dynamics: gas flow through a variable nozzle [7]

4 Pattern formation

- Two-front pulses in a bistable reaction-diffusion system [4]
- Spikes and spike patterns in the Gierer-Meinhardt system [6, Example 1.1], [11, Section 20.6a], [3], [5]

5 Beyond normal hyperbolicity

- Fold points and blowup [9], [11, Section 7.4]
- Blowup, relaxation oscillations and canards [10]

6 Related techniques for asymptotic analysis

- Periodic averaging [18, Chapter 11], [11, Section 9.6]
- Two-timing / Poincaré-Lindstedt [18, Chapter 10], [11, Section 9.8]
- (Wilson) Renormalisation group [11, Sections 9.9-9.10], [2]

7 Mathematical background and proofs

- Proof of Fenichel's main theorem (perturbations of invariant manifolds): [11, Section 2.2]
- Exchange Lemma and relaxation oscillations in the Fitzhugh-Nagumo model [11, Sections 6.2-6.5]

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